Measuring Holes of 3D Meshes

Yann-Situ Gazull   Aldo Gonzalez-Lorenzo   Alexandra Bac

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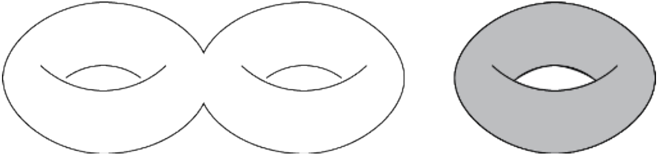
Introduction

Holes

- 0-holes : connected components
- 1-holes : tunnels
- 2-holes : cavities



two 0-holes

Holes

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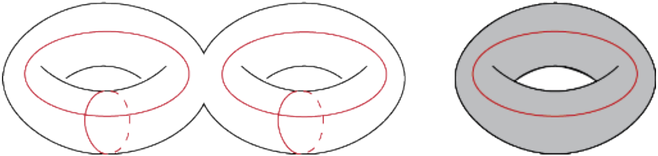

two 0-holes, five 1-holes

Holes

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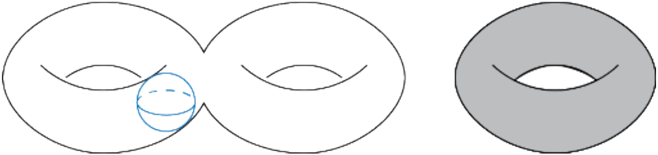
- 1-holes : tunnels
- 2-holes : cavities



two 0-holes, five 1-holes and one 2-hole.

Persistent Homology

Persistent homology keeps track of holes appearing and disappearing in a filtration $(F_t)_{t\in\mathbb{R}}$ as t grows.

Definition (Filtration)

$(F_t)_{t\in\mathbb{R}}$ is a filtration iff

$$t \leq t' \implies F_t \subset F_{t'}$$

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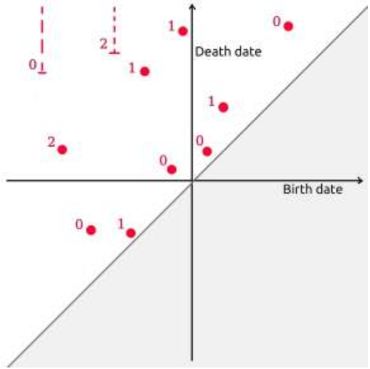
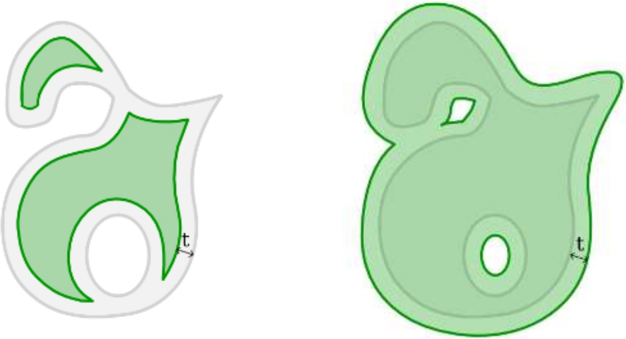


sdf-Filtration

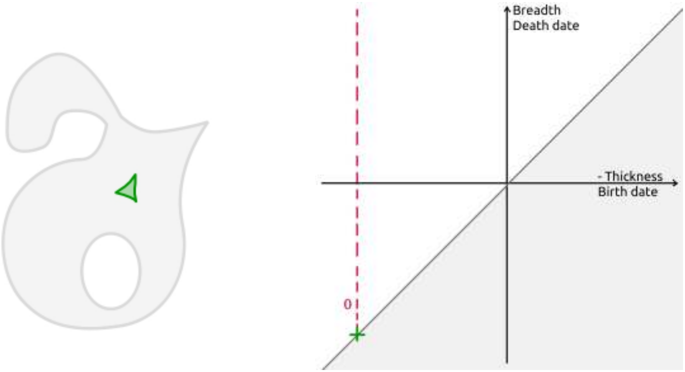
Definition

$$\mathcal{F}_t := sdf^{-1}\left(\left]-\infty, t\right[\right)$$

$(\mathcal{F}_t)_{t\in\mathbb{R}}$ is called the *sdf-filtration*.



sdf-Persistence



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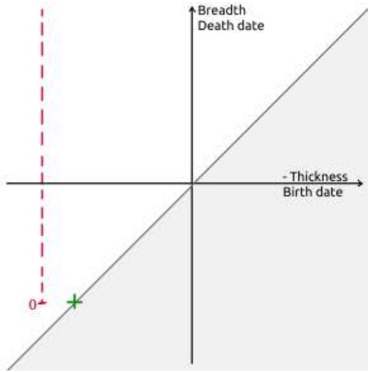

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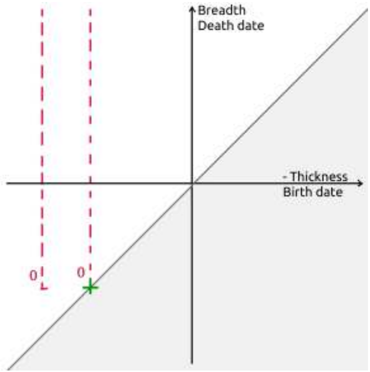

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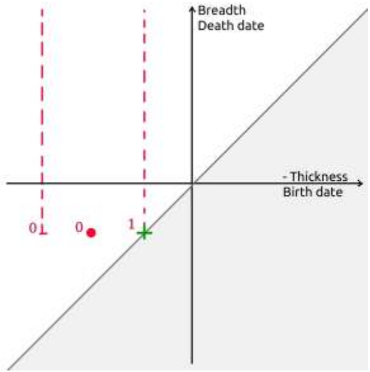

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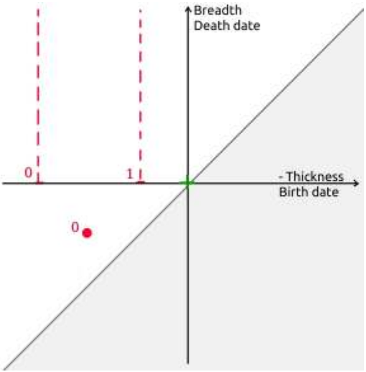

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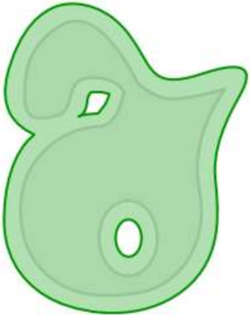

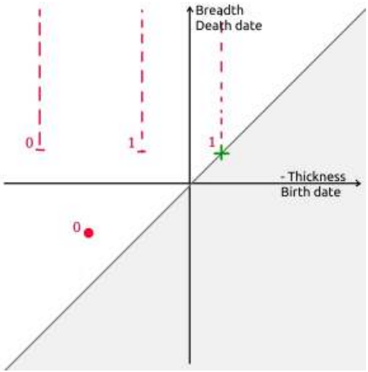

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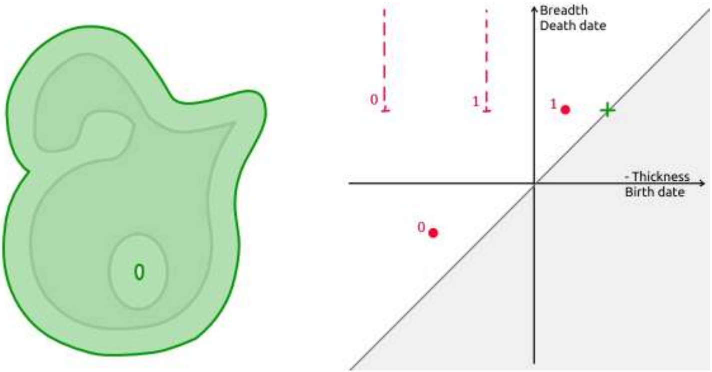

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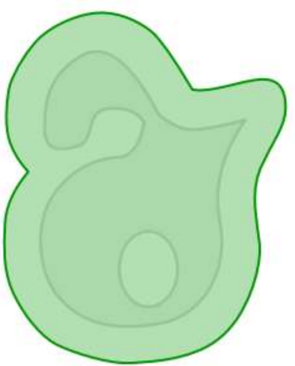
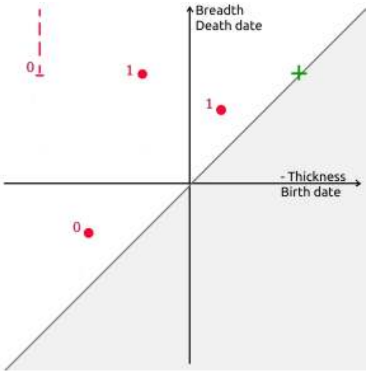
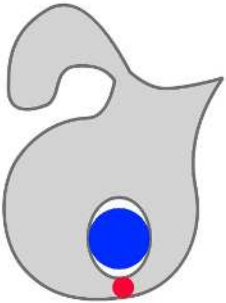
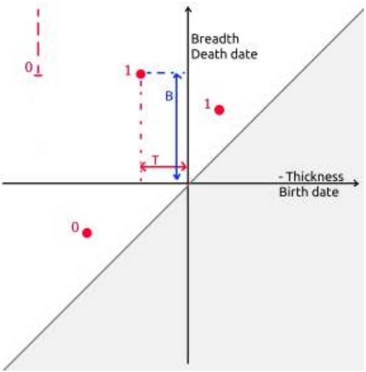


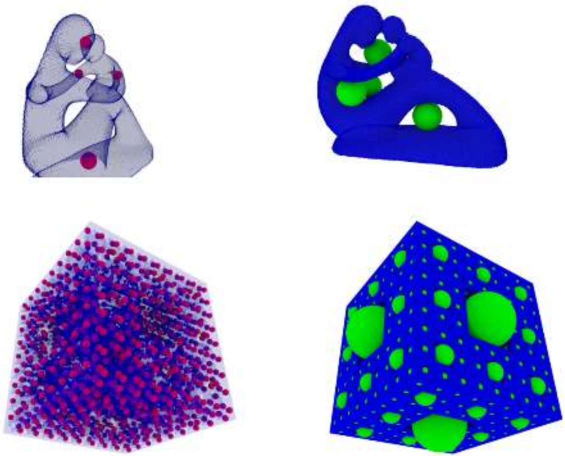


TB-balls





Hole Measures on Cubical Complexes



A. Gonzalez-Lorenzo et al (2016)

Medial Axis

The Medial Axis

Definition (Medial Axis)

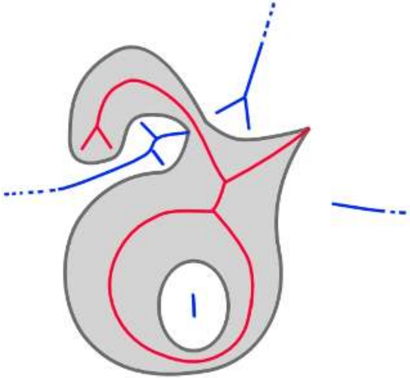
The *medial axis* $\mathcal{M}(X)$ of X is the set of points that have two or more closest points on the boundary of X.

The *inner medial axis* :

$$\check{\mathcal{M}}(X) = \mathcal{M}(X) \cap X$$

The *outer medial axis* :

$$\hat{\mathcal{M}}(X) = \check{\mathcal{M}}(\mathbb{R}^n \backslash X)$$



Properties

Theorem (Lieutier (2003))

For all bounded open X:

$$\check{\mathcal{M}}(X) \approx X$$

Where $\approx$ stands for homotopy equivalence.

Measuring Holes of 3D Meshes Using Medial Axes

Overview

- Persistence of X on $]-\infty, 0]$ (and T-balls) can be obtained from persistence of $\check{\mathcal{M}}(X)$.
- Persistence of X on $[0, +\infty[$ (and B-ball) can be deduced from persistence of $\hat{\mathcal{M}}(X)$ mapped to the sphere S^n. [Conjecture]

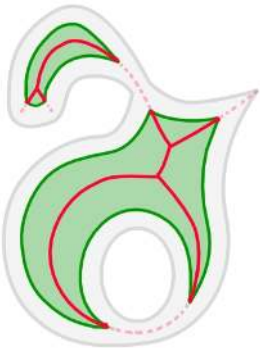
T-balls from the inner medial axis

Theorems

Theorem

Given X an open bounded set of $\mathbb{R}^n$ and $t \geq 0$:

$$\mathcal{F}_{-t} \cap \check{\mathcal{M}}(X) = \check{\mathcal{M}}\Big(\mathcal{F}_{-t}\Big)$$



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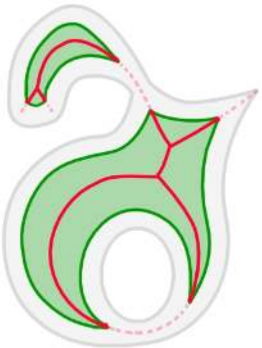


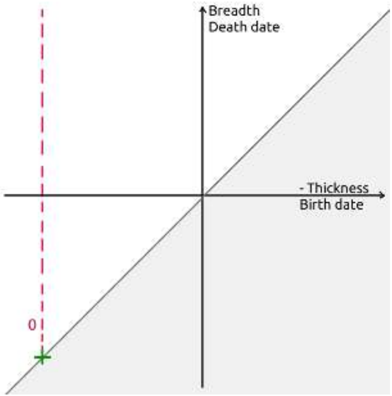
Theorems

Theorem

Given X an open bounded set of $\mathbb{R}^n$ and its associated $\mathcal{F}_t$:

$$\mathcal{D}\left((\mathcal{F}_t)_{t\in]-\infty,0]}\right) = \mathcal{D}\left(\left(\mathcal{F}_t \cap \check{\mathcal{M}}(X)\right)_{t\in]-\infty,0]}\right)$$

Illustration



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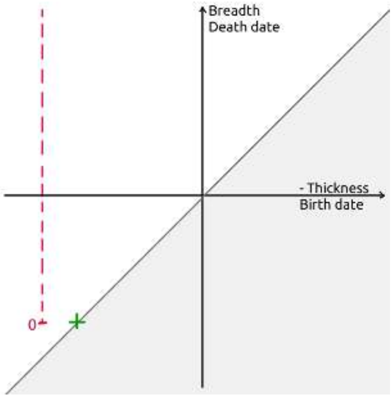

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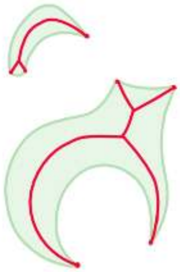

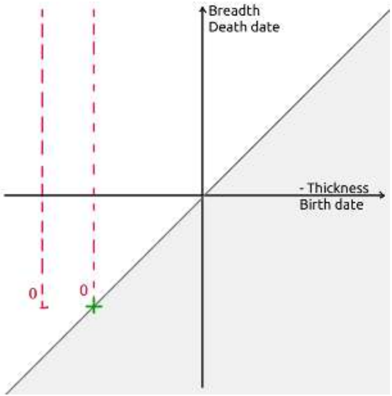

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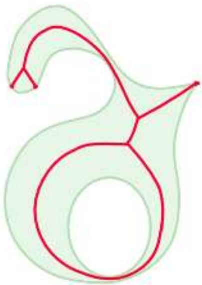

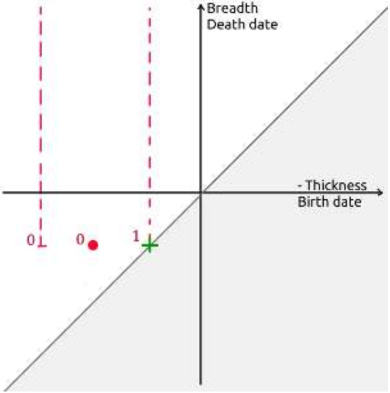

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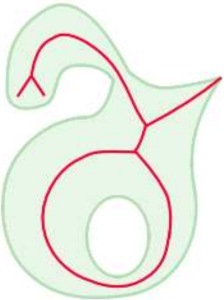

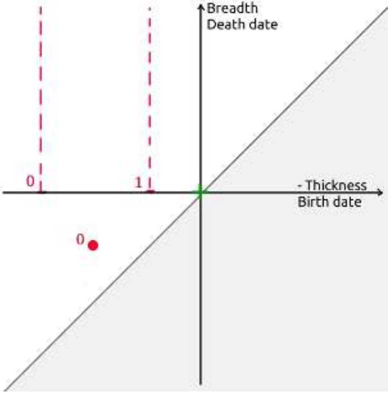

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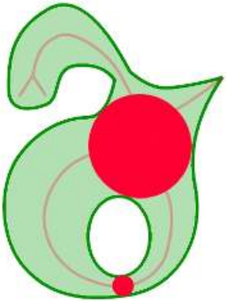
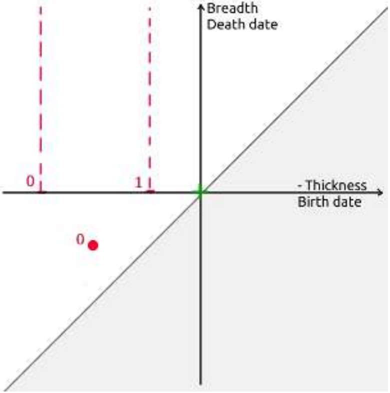




B-balls from the outer medial axis

Alexander Duality

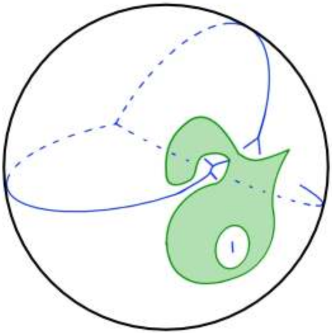
Theorem (Alexander Duality)

Each i-hole in X corresponds to a $n - i - 1$*-hole in* $S^n \backslash X$*, except for a 0-hole, which corresponds to a 0-hole in* $S^n \backslash X$*.*

Conjectures

Main Conjecture

Persistence of X on $[0, +\infty[$ can be deduced from persistence of **its outer medial axis** $\hat{\mathcal{M}}(X)$ mapped to S^n.



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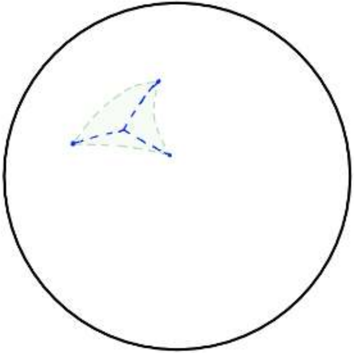
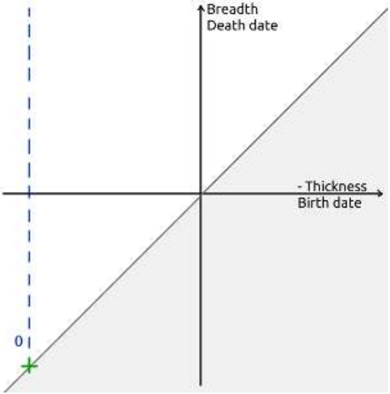
- Persistence of X on $[0, +\infty[$ can be deduced from persistence of $S^n \backslash X$ on $]-\infty, 0]$ using **Alexander duality**.

- Persistence of $S^n \backslash X$ on $]-\infty, 0]$ can be obtained from persistence of $\hat{\mathcal{M}}(X) = \check{\mathcal{M}}(S^n \backslash X)$ mapped to S^n.

Complementary Holes





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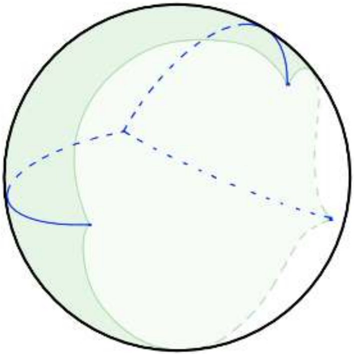

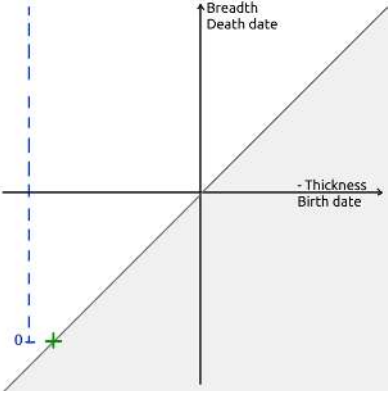

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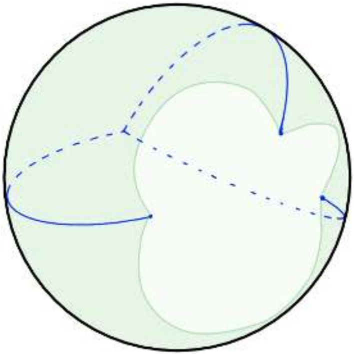

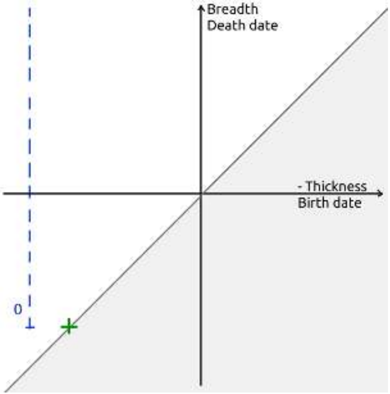

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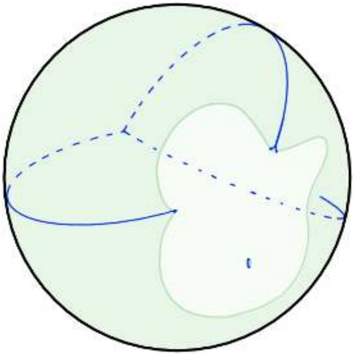

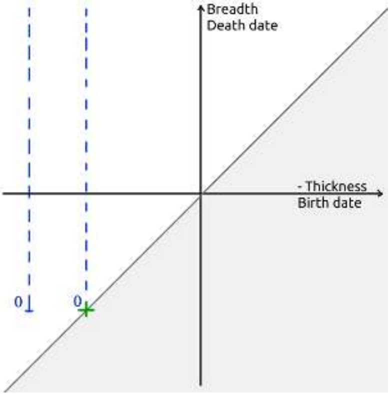

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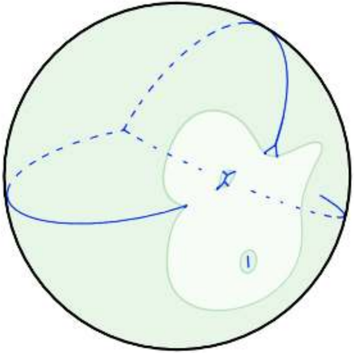

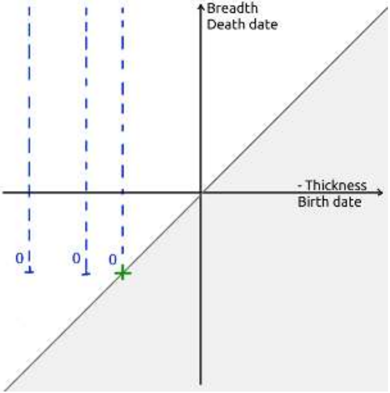

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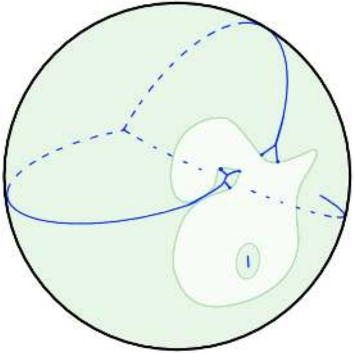

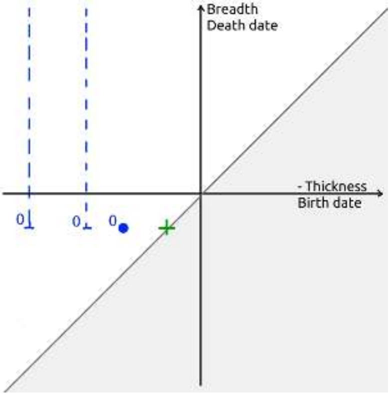

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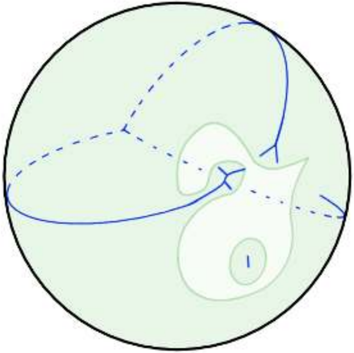
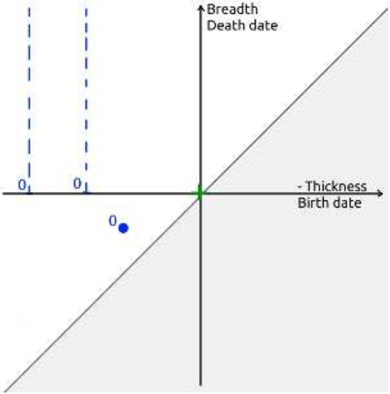
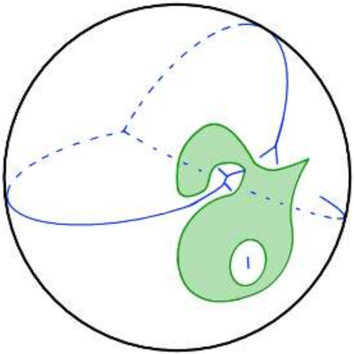
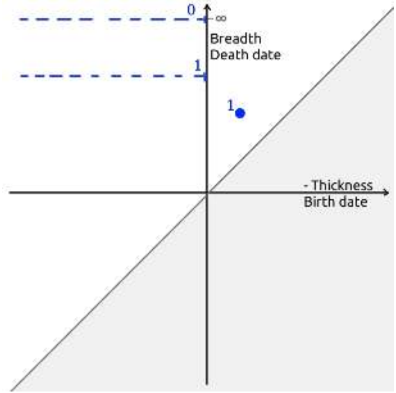




Alexander Deduction





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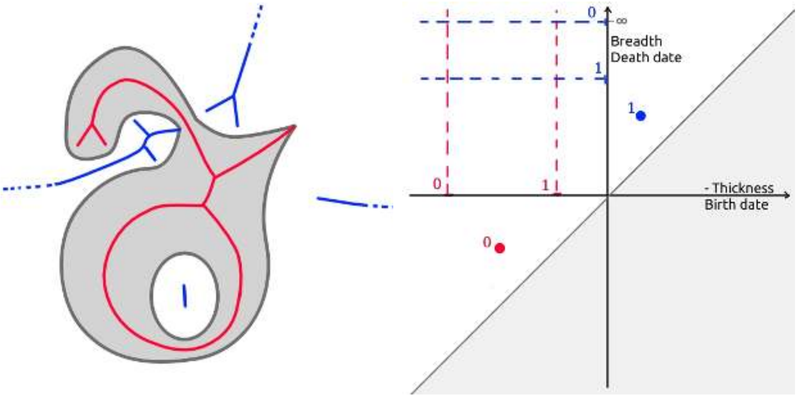
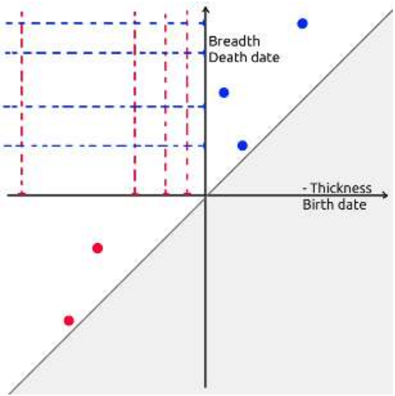


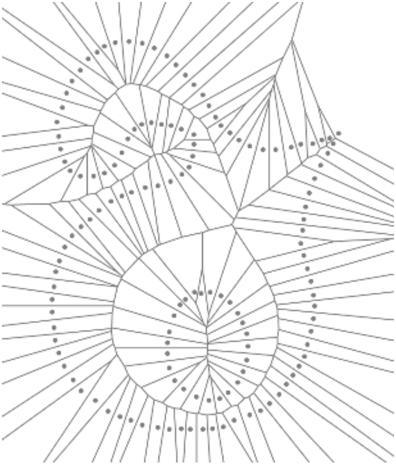
Partial Persistence



Prospects

Medial Axes using Voronoï Diagrams

- Giensen (2011)
- Cazals (2008)
- K.Dey (2004)



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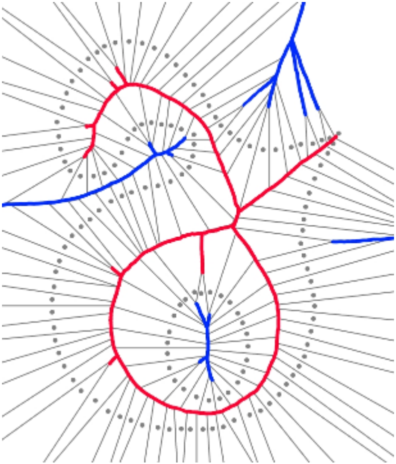
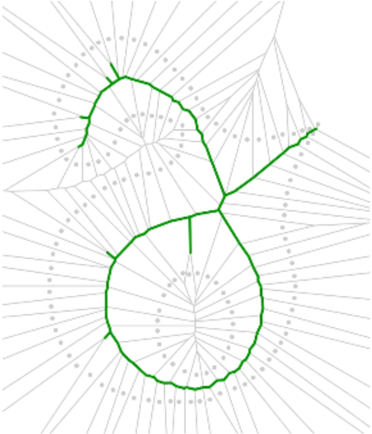


Full Persistence using Voronoï Filtration



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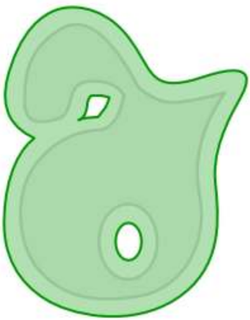

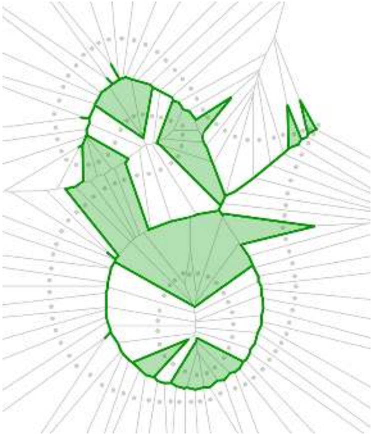

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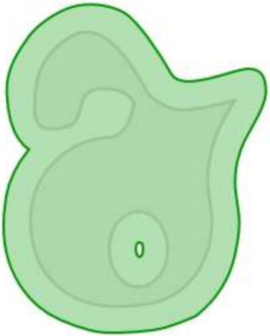

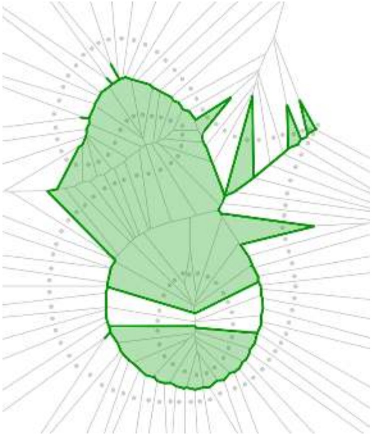

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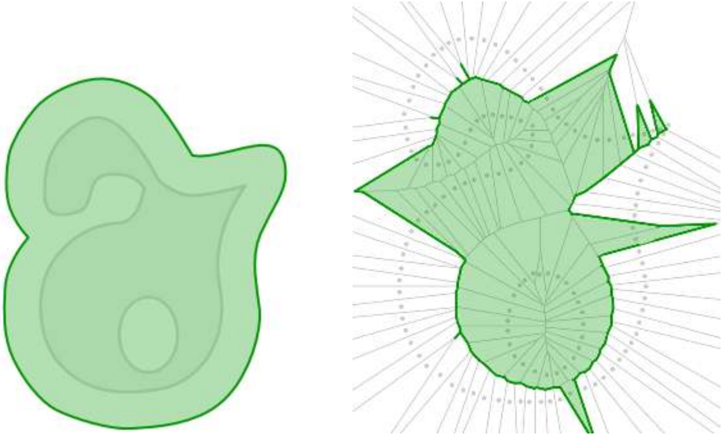

Conclusion

- Every hole has two independent measures that can be represented using balls.
- Persistence on medial axes of X provide partial persistence and every TB-ball of X.